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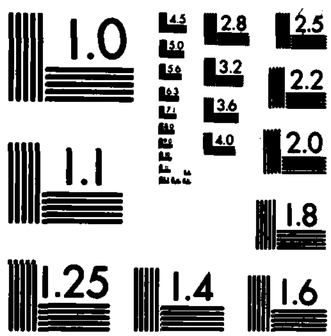

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MALLIAVIN'S CALCULUS AND STOCHASTIC INTEGRAL REPRESENTATIONS OF FUNCTIONALS OF DIFFUSION PROCESSES

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ABSTRACT

If F is a Fréchet differentiable functional on $C[0,1]$, $b = \{b(t)|0 < t < 1\}$ is a Brownian motion, and $\underline{B}_t = \sigma\{b(s)|s < t\}$, Clark's formula states that $F(b) = \int_0^1 E\{\lambda^F(s,1];\ \cdot)|\underline{B}_s\}db(s)$, where $\lambda^F(du;b)$ is the measure defining the Fréchet derivative of F at b. In this paper we extend Clark's formula to the more general class of weakly H-differentiable functionals, and we give a simple proof based on Malliavin's calculus. Again using Malliavin calculus techniques, we also derive Haussmann's stochastic integral representation of a functional $F(y)$ of the diffusion process $dy = m(t,y)dt + \sigma(t,y)db$. In doing this, we show that $y(t)$ is weakly H-differentiable if m and σ have bounded, continuous, first derivatives in y.

AMS (MOS) Subject Classifications: 60H05, 60H10

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SIGNIFICANCE AND EXPLANATION

In stochastic analysis, we often encounter functionals $F(y(s);\ 0 \leq s \leq t)$ of a diffusion process $y(\cdot)$; e.g., the solution of a stochastic differential equation is a functional of its stochastic input; an estimate $E\{x(t)|y(s),\ 0 \leq s \leq t\}$ of a random process $x(t)$ based on observing $y(s),\ 0 \leq s \leq t$, is a functional of $y(\cdot)$. A theory of such functionals making essential use of the randomness in $y(\cdot)$ is therefore of interest. For example, it is possible, and useful, to find fairly explicit representations of such functionals by stochastic integrals, and formulas of Clark and Haussmann give such representations in the cases that F is Fréchet differentiable (plus, technical conditions) and either y is Brownian (Clark) or y is an Ito process (Haussmann).

The recent invention of the so-called Malliavin calculus has also led to new advances in the analysis of functionals of Brownian motion. Basically, the Malliavin calculus is a method for integrating by parts in function space and with respect to Wiener measure. One version of the theory can be developed through the use of the Clark-Haussmann formulas (Bismut). Another approach uses a second-order, self-adjoint operator on functionals and the natural concept of differentiation in Wiener space, the H-derivative. Let $H = \{\gamma|\gamma(t) = \int_0^t \gamma'(s)ds,\ \int_0^1 (\gamma'(s)^2)ds < \infty\}$. Then if $b(\cdot)$ is a Brownian motion, $b(\cdot) + \gamma(\cdot)$ generates a measure absolutely continuous w.r.t. Wiener measure iff $\gamma \in H$. Hence it makes sense to consider only derivatives in H directions, $DF(b)\cdot\gamma = \frac{d}{ds} F(b+s\gamma)\big|_{s=0}$, where $\gamma \in H$.

In this paper, ~~we~~ show that this second form of Malliavin's calculus leads to a very simple derivation of Clark's integral representation of $F(b(\cdot))$, where $b(\cdot)$ is Brownian, and, at the same time, extend the result to the broader and more natural class of H-differentiable functionals. This demonstrates the equivalence of the two approaches to Malliavin's calculus and leads to a nice interpretation of Clark's formula. We then use Malliavin calculus techniques to rederive Haussmann's representation of $F(y)$ if y is a diffusion process. In doing this we show under fairly weak smoothness conditions on the diffusion coefficients of $y(\cdot)$, that $y(t)$ must itself be H-differentiable.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MALLIAVIN'S CALCULUS AND STOCHASTIC INTEGRAL REPRESENTATIONS OF FUNCTIONALS OF DIFFUSION PROCESSES

Daniel Ocone*

1. Introduction

Let $\{b(t) | 0 \leq t \leq 1\}$ be a standard, Brownian motion and $\underline{B}_t = \sigma\{b(s) | 0 \leq s \leq t\}$ the filtration it generates. Suppose $F(b)$ is a functional on Brownian paths, and $E[F^2(b)] < \infty$. Then, according to martingale representation theory, there is a $\underline{B}$-adapted process $f(t)$ such that $E[F(\cdot)|\underline{B}_t] = \int_0^t f(s)db(s)$, almost surely, for every $0 \leq t \leq 1$. In [2], Clark showed that if F is Fréchet differentiable and satisfies certain technical regularity conditions, then

$$f(t) = E\{\lambda^F((s,1]; \cdot)|\underline{B}_t\} \quad \text{a.s.}$$

for each $0 \leq t \leq 1$, where $\lambda^F(du;b)$ denotes the signed measure associated to the Fréchet derivative $dF(b)$. As a consequence

$$F(b) = \int_0^1 E[\lambda^F((s,1]; \cdot)|\underline{B}_s]db(s) \quad . \tag{1.1}$$

In [5], Haussmann extended this formula to functionals $F(y(\cdot))$ of processes $y(t)$ satisfying

$$dy = m(t,y)dt + \sigma(t,y)db, \; y(0) = y_0 \in \mathbb{R} \tag{1.2}$$

where $m(t,y)$ and $\sigma(t,y)$ are causal functionals of $y(\cdot)$. Since $y(t) = y(t,b)$, $G(b) = F(y(\cdot,b))$ defines a Brownian functional, and, roughly speaking, to find a representation for $F(y)$ one applies (1.1) to G. Ignoring hypothesis on m, σ and F, Haussmann's result states that

$$F(y) = \int_0^1 E\{\int_{(s,1]} \lambda^F(du;y)Z(u)Z^{-1}(s)|\underline{B}_s\} \, \sigma(s,y)db(s) \tag{1.3}$$

in which Z solves the equation of first variation associated to (1.2), (see (4.5)). (1.1) and (1.3) have appropriate versions for multi-dimensional y and b.

Other proofs of (1.1) and (1.3) than those originally given by Clark and Haussmann have become available. Davis [3] shows that the form of (1.3) arises quite naturally from potential theoretic arguments. Haussmann [6] and Bismut [1] recover these formulas neatly

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by using a Girsanov transformation, and, in Bismut's case, results on stochastic flows. These alternate approaches do not significantly generalize the conditions on F, b, and σ.

Bismut [1] also contains a significant application of (1.3). He uses (1.3) as the basis for an alternative development of the Malliavin calculus. The Malliavin calculus, basically a theory of integration by parts for functionals on Wiener space, can also be derived by introducing a self-adjoint operator which acts on square integrable, Wiener functionals and which is the infinite dimensional analogue of the Ornstein-Uhlenbeck generator (Malliavin [8], Shigekawa [11], Stroock [9, 10]). [1] shows that the Haussmann formula in effect achieves an integration by parts.

It is of interest, therefore, to determine whether the Clark and Haussmann formulas are more general than the Malliavin calculus, or, conversely, whether the operator version of the calculus leads to these formulas. In this paper, we resolve this issue by using the Malliavin calculus, in its manifestation due to Stroock [9] and Shigekawa [11], to prove (1.1) and (1.3). The exercise contains several points of interest. First we find that Clark's formula, (1.1), *is a simple*, immediate consequence of the most basic properties of the Malliavin calculus, and so we obtain a nice explanation of its form. Second, we identify what seems to be the proper class of functionals F for which to frame Clark's formula (see theorem (3.1)). These are the weakly H-differentiable functionals ([11]), i.e. functionals that are differentiable in a weak, Sobolev sense in the direction of any absolutely continuous function. Our formulation explains the technical conditions placed on F in previous statements of Clark's formula; they insure that F be weakly H-differentiable. Section 2 previews H-differentiability and those elements of the Malliavin calculus needed to prove Clark's formula in section 3.

In section 4, we prove Haussmann's formula, again using Malliavin calculus results, but restricting the treatment to coefficients m and σ such that $m(t,y) = m(t,y(t))$ and $\sigma(t,y) = \sigma(t,y(t))$. To do so requires that we prove the weak H-differentiability of

$y(t)$ under weaker conditions on m and σ than have been previously considered (see [9, 10] and [11]). This is done in theorem (4.14), in which it is shown that bounded continuity of the y-derivatives of m and σ suffices for weak H-differentiability.

2. Differential Calculus in Wiener Space

This section gives a brief resumé of the Malliavin calculus as presented in Stroock [9] and of the notion of weak H-differentiability (Shigekawa [11]) and its connection to Stroock's set up. The following notation shall be used in the rest of the paper. $(B, \underline{B}, \mu, \underline{B}_t)$ will denote d-dimensional Wiener space with the standard filtration, that is, $B = \{b \in C([0,T]; R^d) | b(0) = 0\}$, $\underline{B}_t = \sigma\{b(s) | 0 \leq s \leq t\}$, $\underline{B} = \underline{B}_1$, and μ = Wiener measure on $(B, \underline{B})$. $\|b\| = \sup_{[0,1]} |b(t)|$ will denote the sup norm on B. We shall also use the Hilbert space

$$H = \{\gamma \in B \mid \gamma \text{ is abs. cont. and } \int_0^1 \langle\gamma'(s),\gamma'(s)\rangle ds < \infty\}$$

equipped with the inner product $\langle\gamma_1,\gamma_2\rangle_H = \int_0^1 \langle\gamma_1'(s),\gamma_2'(s)\rangle ds$. If $i : H \to B$ is the inclusion map, it is well known that (i,H,B) is an abstract Wiener space and that μ extends the Gauss measure on H. We shall be concerned with derivatives of functionals $F : B \to R$. The Fréchet derivative of F will be written $dF(b)$, or, in its guise as a d-vector of signed, Borel measures $\lambda^F(ds;b) = (\lambda_1^F(ds;b),\ldots,\lambda_d^F(ds;b))$. In other words $dF(b)\cdot u = \int_0^1 \langle u(s),\lambda^F(ds;b)\rangle$ for $u \in B$.

The derivative appropriate to Wiener space is not the Fréchet derivative but the H-derivative.

<u>(2.1) Definition:</u> If $DF(b)$ is an element of H such that

$$|F(b+\gamma) - F(b) - \langle\gamma,DF(b)\rangle_H| = o(|\gamma|_H)$$

for every $\gamma \in H$, F is said to be H-differentiable at b and $DF(b)$ is called its H-derivative.

If F is Fréchet differentiable and $\gamma \in H$,

$$dF(b)\cdot\gamma = \int_0^1 \langle\gamma(s),\lambda^F(ds;b)\rangle = \int_0^1 \langle\gamma'(s),\lambda^F((s,1];b)\rangle ds \quad .$$

It follows immediately that F is then H-differentiable and

$$(2.2) \qquad DF(b)(t) = \int_0^t \lambda^F((s,1];b)ds \quad .$$

Shigekawa [11] also introduces a notion of weak H-differentiability as follows.

<u>(2.3) Definition:</u> If $F(b) = f((\ell_1,b),\ldots,(\ell_n,b))$ where $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable and $\ell_1,\ldots,\ell_n \in B^*$, F is called a <u>cylinder function</u>. F is a <u>smooth cylinder function</u> if $f \in C_0^\infty(\mathbb{R}^n)$.

Let $|F|_p = E^{1/p}|F|^p + E^{1/p}|DF|_H^p$ and $\mathring{H}(p) = \{F | F$ is a Fréchet diff. cylinder function, $|F|_p < \infty\}$.

<u>(2.4) Definition:</u> a) $H(p) :=$ completion of $\mathring{H}(p)$ with respect to $|\cdot|_p$.

b) $H^{(\infty)} = \bigcap_{p>1} H(p)$.

If $F \in H(p)$, a sequence $\{F_n\}$ of differentiable cylinder functions exists such that DF_n is Cauchy in $L^p(B,\mu;H)$. Thus there is a $(L^p\text{-})$ convergent subsequence DF_{n_k} and we make the

<u>(2.5) Definition:</u> $DF = \lim_{k\to\infty} DF_{n_k}$. DF is called the weak H-derivative of F.

(2.6) <u>Lemma</u> (Shigekawa [11])

i) DF in (2.5) is well defined.

ii) Smooth cylinder functions are dense in $H(p)$, $\forall p > 1$.

One may readily verify that $H(p)$ is a separable, reflexive Banach space for $p > 1$ and hence that $|\cdot|_p$ bounded subsets of $H(p)$ are relatively (sequentially) weakly compact. This leads to a useful criterion that $F \in H(p)$. Suppose $F_n \in H(p)$ $\forall n$ and

$$(2.7) \qquad \begin{array}{l} \text{i) } \lim E|F-F_n|^q = 0 \quad , \quad \text{some } q > 1 \\ \text{ii) } \sup_n |F_n|_p < \infty \quad . \end{array}$$

Then $F \in H(p)$.

Our definitions so far have introduced a small ambiguity. DF is used to denote both the H-derivative and the weak H-derivative, although it is possible for $DF(b)$ to be defined for all B in the sense of (2.1) but not in the sense of (2.5). In deriving the usual form of Clark's formula, we shall need to identify the two under certain circumstances.

(2.8) Lemma. Let $F : B \to R$ satisfy

i) F is continuous and H-differentiable (as in (2.1))

ii) $DF(b)$ is strongly measurable ($DF(b)$ = H-derivative of (2.1))

iii) There exist positive constants K, a such that

$$|F(b)| + |DF(b)|_H \le K(1 + \|b\|^a) \quad .$$

Then $F \in H(2)$ and the weak H-derivative coincides with DF μ-almost surely.

Proof. Let $\{T^{(n)}\}$, where $T^{(n)}$ is given by $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = 1$, be a sequence of partitions that becomes dense in $[0,1]$ as $n \to \infty$. If $b \in B$, let

$$b^{(n)}(t) = \begin{cases} b(t_i^{(n)}) & t = t_i^{(n)} \\ \dfrac{1}{t_{i+1}^{(n)}-t_i^{(n)}} \left[(t_{i+1}^{(n)}-t)b(t_i^{(n)}) + (t-t_i^{(n)})b(t_{i+1}^{(n)})\right] & \text{if } t_i^{(n)} < t < t_{i+1}^{(n)} \end{cases}$$

Likewise define $F_n(b) = F(b^{(n)})$. For each n, F_n is Fréchet differentiable and $\langle DF_n(b),\gamma\rangle_H = \langle DF(b^{(n)}),\gamma^{(n)}\rangle_H$. Also, because $\lim_{n\to\infty} \|b-b^{(n)}\| = 0$ and $\|b^{(n)}\| \le \|b\|$ for every $b \in B$, it is clear from assumptions i) and iii) and dominated convergence that $\lim_{n\to\infty} E(F-F_n)^2 = 0$. Thus, to prove that $F \in H(2)$, it is enough to show that $\sup_n |F_n|_2 < \infty$ (see (2.7)). However, since $\langle DF_n(b),\gamma\rangle_H = \langle DF(b^{(n)}),\gamma^{(n)}\rangle_H$, (iii) implies that $|F_n(b)|^2 + |DF_n(b)|_H^2 \le |F(b^{(n)})|^2 + |DF(b^{(n)})|_H^2 \le [K(1 + \|b\|^a)]^2$. It follows immediately that $\sup_n |F_n|_2 < \infty$.

Let $\eta(b)$ be the weak H-derivative of F. It remains to show that $\eta(b) = DF(b)$ μ-a.s. where $DF(b)$ is defined as in (2.1). For this it suffices to show that for each $\gamma \in H$ and $t > 0$,

$$(2.9)\quad \begin{aligned} E\{[F(b+t\gamma) - F(b)]G(b)\} &= E[G(b) \textstyle\int_0^t \langle DF(b+s\gamma),\gamma\rangle ds] \\ &= E[G(b) \textstyle\int_0^t \langle \eta(b+s\gamma),\gamma\rangle_H ds] \end{aligned}$$

for each $G \in L^\infty(B,\mu)$. Indeed, if (2.9) is true, then

$$(2.10)\quad \langle DF(b+s\gamma),\gamma\rangle_H = \langle \eta(b+s\gamma),\gamma\rangle_H$$

for all $\gamma \in H$, for (Lebesgue) almost all s, μ-a.s. Let $\mu_{s\gamma}$ = measure induced on B by $\beta(\cdot) + s\gamma(\cdot)$ where β is a standard, R^d-valued Brownian motion. Since $\mu_{s\gamma} \sim \mu$, (2.10) implies that $\langle DF(b),\gamma\rangle_H = \langle \eta(b),\gamma\rangle_H$ for all $\gamma \in H$, μ-a.s., or, in other words, that $DF(b) = \eta(b)$ μ-a.s. To establish (2.9), begin by noting that F is the weak limit in $H(2)$ of a subsequence $\{F_{n_k}\}$. In particular, for every $G \in L^\infty(B,\mu)$ and every $\gamma \in H$, $\lim_{k\to\infty} E\langle DF_{n_k}(b),\gamma G(b)\rangle_H = E\langle \eta(b),\gamma G(b)\rangle_H$. Likewise, since $\frac{d\mu_{s\gamma}}{d\mu} \in L^2(B,\mu)$ uniformly for fixed γ on compact subsets of s,

$$\lim_{k\to\infty} E\langle DF_{n_k}(b+s\gamma),\ \gamma G(b)\rangle_H$$

$$= E\langle \eta(b),\ \gamma G(b-s\gamma)\rangle_H \frac{d\mu s\gamma}{d\mu}$$

$$= E\langle \eta(b+s\gamma),\ \gamma G(b)\rangle_H$$

and $\{E\langle DF_{n_k}(b+s\gamma),\ \gamma G(b)\rangle_H\}$ is uniformly bounded on compact subsets of s. Thus, again invoking i) - iii),

$$E[F(b+t\gamma) - F(b)]G(b) = \lim_{k\to\infty} E[F_{n_k}(b+t\gamma) - F_{n_k}(b)]G(b)$$

$$= \lim_{k\to\infty} E \int_0^t \langle DF_{n_k}(b+s\gamma),\ \gamma\rangle_H ds\ G(b)$$

$$= E \int_0^t \langle \eta(b+s\gamma),\ \gamma\rangle_H ds\ G(b)$$

as desired.

□

The Malliavin calculus introduces another differential operator on $L^2(B,\mu)$ which is the analogue of the finite-dimensional Ornstein-Uhlenbeck operator. To define this most directly we first recall the Ito-Wiener decomposition

$$L^2(B,\mu) = \bigoplus_{n=0}^{\infty} I^{(n)}$$

where
$$I^{(n)} = \{\int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} f(s_1,\dots,s_n)d\beta_{i_n}(s_n)\cdots d\beta_{i_1}(s_1),\ 1 \le i_k \le d,$$
$$f \in L^2([0,T]^n,dx)\}$$

is the space of n^{th} order, multiple stochastic integrals. Alternatively stated, if $F \in L^2(B,\mu)$ and $P^{(n)} := \mathrm{Proj}|I^{(n)}$

$$F = \sum_{n=0}^{\infty} P^{(n)}F \quad .$$

(2.11) Definition

$$AF = -\sum_{n=0}^{\infty} \frac{n}{2} P^{(n)}F$$

if $F \in D_2 = D(A)$,

$$D(A) = \{F \in L^2(B,\mu) | \sum_0^{\infty} n^2 E(P^{(n)}F)^2 < \infty\} \quad .$$

It turns out (see [9]) that A is a non-positive definite, self-adjoint operator that generates an Ornstein-Uhlenbeck type Markov semigroup. Moreover, if $G(p) = D_2 \cap \{E[|F|^p + |AF|^p] < \infty\}$, $A|G_p$ extends consistently to a closed operator A_p on $L^p(B,\mu)$ for $p > 1$; that is, $A_p|D(A_q) = A_q$ if $q > p$. Without causing any ambiguity, we shall drop the subscript p.

(2.12) Lemma [9] If $F, G \in D_2$, then $F \cdot G \in D_1$.

(2.13) Definition. Let $F, G = D_2$.

$$\nabla F \cdot \nabla G = AFG - F(AG) - G(AF) \quad .$$

A and $\nabla F \cdot \nabla G$ are the basic tools of the Malliavin calculus. The following theorem collects some basic facts about their use.

(2.14) Theorem [9]

a) Let $F, G \in D_2$ and assume i), F is $\underline{B}_t$-measurable, ii) G is $\sigma\{b(u) - b(t) | t \le u \le 1\}$-measurable. Then

$$A(FG) = F(AG) - G(AF) \quad .$$

b) If $F, G \in D_2$, then $E\nabla F \cdot \nabla G = -2EFAG$.

Shigekawa [11] defines A by using higher order weak H derivatives. We prefer Stroock's direct approach (2.11) since we shall utilize his applications to stochastic differential equations. However it is important for us to connect D_2 to weak H-differentiability. The notation $\nabla F \cdot \nabla G$ introduced in (2.13) begs an analogy to a gradient inner product. The next result makes this precise.

(2.15) Theorem

i) $D_2 \subset H(2)$

ii) If $F, G \in D_2$, then $\nabla F \cdot \nabla G = \langle DF, DG\rangle_H$.

Proof. Let $F = \sum_0^N P^{(n)}F$, $G = \sum_0^M P^{(n)}G$ where $N, M < \infty$. Then results of Shigekawa [11] directly imply $F, G \in H(2)$ and $\nabla F \cdot \nabla G = \langle DF, DG\rangle_H$. For general $F \in D_2$, let $F_N = \sum_0^N P^{(n)}F$. Then

$$\lim_{N\to\infty} E[|F_N - F|^2 + |AF - AF_N|^2] = 0 \tag{2.16}$$

$$\lim_{N\to\infty} E|\nabla F_N \cdot \nabla F_N - \nabla F \cdot \nabla F| = 0 \quad . \tag{2.17}$$

(2.16) is immediate and (2.17 is proved in [9]. Since

$$\begin{aligned} E|DF_N - DF_M|_H^2 &= E\nabla(F_N - F_M)\cdot\nabla(F_N - F_M) \\ &= -2E(F_N - F_M)A(F_N - F_M) \quad . \end{aligned}$$

DF_N is Cauchy in $L^2(B,\mu;H)$. Thus $F \in H(2)$. Moreover

$$\begin{aligned} E|\nabla F_N \cdot \nabla F_N - |DF|_H^2| &= E|\ |DF_N|_H^2 - |DF|_H^2| \\ &< [\sup_n [E^{1/2}|DF_N|_H^2 + E^{1/2}|DF|_H^2]E[|D(F_N - F)|_H^2] \quad . \end{aligned} \tag{2.18}$$

(2.17) and (2.18) together imply $\nabla F \cdot \nabla F = \langle DF, DF\rangle_H$. The general statement (ii) follows from the polarization identity.

□

The following extension of (2.14) a) will be useful later.

(2.19) Corollary. If $F, G \in H_2$ and F is $\underline{B}_t$-measurable and G is $\sigma\{b(u) - b(t) | t < u < 1\}$-measurable. Then $\langle DF, DG\rangle_H = 0$.

Pf. Approximate F and G by smooth, cylinder functions.

□

3. Clark's formula

In this section we will prove the following theorem and show that it is a general version of Clark's formula.

(3.1) Theorem. Let $F \in H(2)$ and let $\alpha(t)$, $0 \le t \le 1$ be any bounded, R^d-valued, $\underline{B}_t$-adapted, measurable process. Then

$$E\{F(b) \int_0^1 \langle\alpha(s), db(s)\rangle\} = E\{\langle DF(b), \int_0^{\cdot} \alpha(s)ds\rangle_H\}$$

$$= E\{\int_0^1 \langle[DF]'(s), \alpha(s)\rangle ds\} \quad .$$

In (3.1), as in Clark's formula, we equate an expression involving F to one involving DF. In fact, using the following corollary of (3.1), we may easily derive previous statements of Clark's formula.

(3.2) Corollary. Assume

i) F is H-differentiable (as in (2.1)) for each $b \in B$

ii) DF is H-measurable

iii) There exist positive constants K and a such that

$$|F(b)| + |DF(b)|_H \le K(1 + |b|^a) \quad .$$

Then

(3.3) $$F(b) = \int_0^1 \langle E\{[DF(\cdot)]'(s)|\underline{B}_s\}, db(s)\rangle \quad \mu\text{-a.s.}$$

In particular, if F is also Fréchet differentiable

(3.4) $$F(b) = \int_0^1 \langle E[\lambda^F((s,1];\cdot)|\underline{B}_s], db(s)\rangle \quad \mu\text{-a.s.}$$

Proof. A simple application of martingale representation theory (see, eg. [7]) shows that (3.3) is equivalent to

(3.5) $$E\{F(b) \int_0^1 \langle\alpha(s), db(s)\rangle\} = E\{\int_0^1 \langle[DF(b)]'(s), \alpha(s)\rangle ds\}$$

for all bounded, adapted processes α. However, according to lemma (2.8), $F \in H(2)$ and its weak H-derivative is DF. (3.5) is then immediate from theorem (3.1). (3.4) follows from (3.3) because (see (2.2)) $[DF(s)]'(s) = \lambda^F((s,1];b)$ if F is Fréchet differentiable.

□

The identity (3.4) was first proved by Clark [2] for functionals F that are Fréchet differentiable and for which the remainder $R(b_1,b_2) = F(b_1+b_2) - F(b_1) - dF(b_1)\cdot b_2$ satisfies $|R(b_1,b_2)| \le M\|b_2\|^{1+\delta}(1 + \|b_1\|^{\alpha})(1 + \|b_2\|^{\alpha})$ for some positive constants M, δ, and α. Davis [3] requires that F be Fréchet differentiable, $\lambda^F(\cdot;b)$ be weakly continuous in b, and $|F(b)| + \|dF(b)\|_* \le K(1 + \|b\|^a)$ for some K, $a > 0$ where $\| \ \|_* =$ total variation norm of λ^F. Both of these conditions imply hypothesis i) - iii) in corollary (3.2), and hence require that $F \in H(2)$. Thus, the condition that $F \in H(2)$ is more general, and, as will appear from the proof, the theoretically natural one for which to state Clark's formula. Theorem (3.1) thus explains the conditions of Clark and Davis; they guarantee the weak H-differentiability of F.

To prove theorem (3.1) we first establish the following lemma, which is a direct consequence of basic properties of the Malliavin calculus.

(3.6) Lemma. Let α be a smooth $\mathbb{R}^d$-valued cylinder function and assume α is $\underline{B}_t$-measurable. Let $\tau > t$. If $F \in H(2)$, then

$$E\{\sum_1^d \alpha_i \langle DF, D(b_i(\tau) - b_i(t))\rangle\} \tag{3.7}$$
$$= E\{F\langle\alpha, b(\tau) - b(t)\rangle\} \ .$$

Remark. $\frac{d}{ds} D(b_i(\tau) - b_i(t))(s) = 1_{(t,\tau]}(s)e_i$ where e_i is the standard basis vector with 1 in the i^{th} position. Thus

$$\sum_1^d \alpha_i\langle DF, D(b_i(\tau) - b_i(t))\rangle_H = \int_0^1 \langle [DF]'(s), \alpha\, 1_{(t,\tau]}(s)\rangle ds \ . \tag{3.8}$$

Proof of (3.6). It suffices to prove (3.6) when F is a smooth cylinder function also, because, if $F \in H(2)$, (2.6) guarantees the existence of a sequence $\{F_n\}$ of smooth cylinder functions such that $\lim_{n\to\infty} |F_n - F|_2 = 0$. Since

$$|E\{(F-F_n)\langle\alpha, b(\tau) - b(t)\rangle\}|$$

and

$$\le E^{1/2}\{(F-F_n)^2\}E^{1/2}\{\langle\alpha,\alpha\rangle(\tau-t)\}$$

$$|E[\alpha_i \langle D(F-F_n),\ D(b_i(\tau) - b_i(t))\rangle_H]|$$

$$\leq E^{1/2}|D(F-F_n)|_H^2\, E^{1/2}[\alpha_i^2(\tau-t)^2] \quad ,$$

limits may be taken on both sides of (3.7) to prove the case $F \in H(2)$. Thus, assume F is smooth and let $\psi_i = b_i(\tau) - b_i(t)$. From (2.14a) $\Lambda\alpha_i\psi_i = \alpha_i\Lambda\psi_i + (\Lambda\alpha_i)\psi_i$, and, by definition $\Lambda\psi_i = -\frac{1}{2}\psi_i$. Therefore, using the self-adjointness of Λ

$$E\{\alpha_i\langle DF, D\psi_i\rangle_H\} = E\{\alpha_i[\Lambda F\psi_i - F\Lambda\psi_i - \psi_i\Lambda F]\}$$

$$= E\{F\psi_i\Lambda\alpha_i - \alpha_i F\Lambda\psi_i - F[\alpha_i\Lambda\psi_i + \psi_i\Lambda\alpha_i]\}$$

$$= -2EF\alpha_i\Lambda\psi_i = EF\alpha_i\psi_i \quad .$$

□

Proof of (3.1). It suffices to prove (3.1) for bounded, adapted simple process. For a general bounded, adapted $\alpha(s)$, let $\alpha_n(s)$ be a sequence of simple process such that

$$\lim_{n\to\infty} E\int_0^1 |\alpha_n(s) - \alpha(s)|^2 ds = 0$$

Then it is a simple matter to show

$$\lim_{n\to\infty} E\{F\int_0^1 \langle\alpha_n(s), d\beta(s)\rangle\} = E\{F\int_0^1 \langle\alpha(s), d\beta(s)\rangle\}$$

$$\lim_{n\to\infty} E\{\langle DF, \int_0^1 \alpha_n(s)ds\rangle_H\} = E\{\langle DF, \int_0^1 \alpha(s)ds\rangle_H\}$$

and so prove the general case.

Now, for simple functions, it is enough to treat the case $\alpha(s) = \alpha 1_{(t,\tau]}(s)$, where α is $\underline{B}_t$ measurable and bounded. Lemma (3.6) and the remark following prove this case if, in addition, α is smooth. To treat $\alpha(s) = \alpha 1_{(t,\tau]}(s)$ when α is not smooth, choose a sequence of smooth, $\underline{B}_t$-measurable, cylinder functions such that $\lim_{n\to\infty} E|\alpha-\alpha_n|^3 = 0$. It may easily be shown that

$$E[F \int_0^1 \alpha 1_{(t,\tau]}(s)ds] = \lim_{n\to\infty} E[F \int_0^1 \alpha_n 1_{(t,\tau]}(s)ds]$$

$$= \lim_{n\to\infty} E[\langle DF, \int_0^1 \alpha_n 1_{(t,\tau]}(s)ds\rangle_H]$$

$$= E[\langle DF, \int_0^1 \alpha_{(t,\tau]}(s)ds\rangle_H] \quad .$$

This completes the proof.

□

4. The Haussmann formula.

Let $y = \{y(t) : 0 \le t \le 1\}$ be the R^N-valued diffusion that solves

$$(4.1)\qquad \begin{aligned} dy(t) &= m(t,y)dt + \sigma(t,y)db \\ y(0) &= y_0 \quad . \end{aligned}$$

We assume $y_0 \in R^N$, $m(t,y) : [0,1] \times R^N \to R^N$ and $\sigma(t,y) : [0,1] \times R^N \to R^N \otimes R^d$ are Borel measurable in (t,y) and continuously differentiable in y for each t, $b(t)$ is d-dimensional, standard Brownian motion, and

$$(4.2)\qquad \sup_{t\le 1, y\in R^N} \left|\frac{\partial m_i}{\partial y_j}\right| < \infty \qquad 1 \le i, j \le N$$

$$\sup_{t\le 1, y\in R^N} \left|\frac{\partial \sigma_{ij}}{\partial y_k}\right| < \infty \qquad 1 \le i, k \le N \ , \quad 1 \le j \le d$$

$$(4.3)\qquad \sup_{t\le 1} |m(t,0)| + \|\sigma(t,0)\| < \infty \ .$$

Note that (4.2) and (4.3) imply that there exists a constant K such that

$$(4.4)\qquad \sup_{t\le 1} |m(t,y)| + \sup_{t\le 1} \|\sigma(t,y)\| \le K(1 + |y|) \quad .$$

Standard existence theorems then guarantee a unique, strong, a.s. continuous solution $y(t)$ of (4.1).

The equation of first variation associated to (4.1) is

$$dZ = \partial_y m(t,y)\cdot Z\,dt + \sum_{i=1}^{d} \partial_y \sigma_i(t,y)\cdot Z\,d\beta_i$$

(4.5)

$$Z(0) = I_N \quad .$$

(In (4.5) σ_i denotes the i^{th} column of σ.) (4.5) has a unique, $\mathbb{R}^N \otimes \mathbb{R}^N$-valued solution $Z(t)$, which is invertible for each t. Indeed, $W(t) = Z^{-1}(t)$ satisfies

$$dW = W\cdot[-\partial_y m(t,y) + \sum_{i=1}^{d} [\partial_y \sigma_i(t,y)]^2]dt$$

(4.6)

$$- W\cdot \sum_{i=1}^{d} \partial_y \sigma_i(t,y)db \;,$$

$$W(0) = I \quad .$$

$Z(t)Z^{-1}(s)$, $0 \le s \le t \le 1$ then serves as a state transition matrix for (4.5).

Let $\| \ \|_*$ denote the total variation for signed, $\mathbb{R}^N$-valued Borel measures. Let $B^{(N)} = C([0,1];\mathbb{R}^N) \cap \{b|b(0) = 0\}$. In this section, we prove

(4.7) Theorem (Haussmann). Let $F : B^{(N)} \to \mathbb{R}$ be Fréchet differentiable and suppose

(4.8)

i) $\lambda^F(b)$ is weakly continuous in b

ii) $|F(b)| + \| \lambda^F(b)\|_* \le K(1 + \|b\|^a)$ for some K, $a > 0$.

Then

(4.9) $E[F(y)\int_0^1 \langle\alpha(s),db(s)\rangle] = E[\int_0^1 \langle\int_{(s,1]} \lambda^F(du;y)Z(u)Z^{-1}(s)\sigma(s,y(s)),\alpha(s)\rangle ds]$

for every bounded $\underline{B}_t$-adapted process $\{\alpha(t) \mid 0 \le t \le 1\}$.

(Remark. In (4.9), λ^F is interpreted as a row vector.)

The condition (4.8) imposed on F is the same as that given in Davis [3]. However, less restrictions are placed on m and σ in the present treatment because we do not rely on potential theoretic results. Haussmann's [5] original statement of the theorem actually allows $m(t,y)$ and $\sigma(t,y)$ to be causal functionals of y, although more stringent regularity conditions are placed on F. We shall indicate below how the proof given here might be extended to deal with such coefficients.

Our strategy for proving (4.7) begins from the observation that $y(\cdot,b)$ is a functional of Brownian paths and, hence, that $F(y)$ defines the Brownian functional $G(b) = F(y(\cdot,b))$. Thus, roughly speaking, to derive (4.9) it is only necessary to show that $G \in H(2)$ and

$$[DG(b)]'(s) = \int_{(s,1]} \lambda^F(du;y)Z(u)Z^{-1}(s)\sigma(s,y(s))$$

and then to apply theorem (3.1). To be more precise, we actually need to show that $y(t;b) \in H(2)$ and to compute $Dy(t)$. In the case that m and σ satisfy (4.2), (4.3) and, in addition, possess slowly growing, continuous, second derivatives with respect to y, Stroock [9, 10] establishes that $y(t) \in D_p$ for every $p > 1$ (see discussion after (2.11)) and $E[\sup_{[0,1]} |y_i(t)|^{2p} + |Ay_i(t)|^{2p} + |\nabla y_i(t)\cdot\nabla_i(t)|^p] < \infty$, $1 \le i \le N$, and Shigekawa [11] contains similar results. Thus when m and σ are C^2, $y(t) \in H(2)$ is certainly true. In theorem (4.14) we extend this analyis by showing that $y(t) \in \bigcap_{p>1} H(p)$ even if the C^2 assumption is dropped, and by calculating $Dy(t)$. The C^2 assumptions in previous work are necessary only to prove the stronger result that $y(t)$ is in the domain of the second order operator A. Since 'D' is a first order operator only continuous differentiability of m and σ is needed for weak H-differentiability of $y(t)$. Theorem (4.14) is the crucial step, and, once it is established, the proof of Haussmann's formula follows easily. Thus, to extend this method to $m(t,y)$ and $\sigma(t,y)$ which depend functionally of y, it would be necessary to generalize theorem (4.14) appropriately and obtain the natural analogue to equation (4.15) for $Dy(t)$.

We begin with some preliminary lemmas. The first concerns how the property, $F \in H_p$, behaves under the transformation $\phi(F)$ for a function ϕ.

(4.10) <u>Lemma</u>. Let $\phi : R^N \to R$ be a C^1 function such that

$$|\phi(x)| + \sum_{i=1}^{n} |\frac{\partial\phi}{\partial x_i}(x)| \le K(1 + |x|^a)$$

for some $K, a > 0$. Let $q > (a+1)$ and $F = (F_1,\ldots,F_n) \in (H(q))^n = H(q) \times\cdots\times H(q)$. Then if $p < q/a+1$, $\phi(F) \in H(p)$ and

$$D\phi(F(b)) = \sum_{1}^{n} \frac{\partial\phi}{\partial x_i}(F(b))DF_i(b) \quad . \tag{4.11}$$

In particular, if $F \in H^{(\infty)}$, $\phi(F) \in H^{(\infty)}$.

Proof. (4.11) is straightforward to establish if ϕ has compact support and F is a smooth cylinder function. If $F \in (H(q))^n$, let $F_k \to F$ in $(H(q))^n$ as $k \to \infty$ where F_k are smooth cylinder functions, and take limits in (4.11) still assuming ϕ to have compact support. When ϕ does not have compact support let $\phi_k(x) = \phi(x)\rho(x/k)$ where $\rho(x) \in C^1$, $\rho(x) = 1$ for $|x| < 1$, $\rho(x) = 0$ for $|x| > 2$ and $0 < \rho(x) < 1$, $\forall x$. Take limits as $k \to \infty$ to achieve the final result. The condition $p < q/a+1$ insures that $E(D(\phi(F))^p < \infty$.

□

The second lemma addresses a similar issue. What can be said about the weak H-differentiabily of the integral $\int_0^t f(s,x(s))ds + \int_0^t \langle a(s,x(s)), db(s)\rangle$ if $x(t)$ is a β-adapted process such that $x(t) \in H^{(\infty)} = \bigcap_p H_p$, $\forall\, t < 1$?

(4.12) Lemma. Let $f(s,x) : [0,1] \times R^N \to R$, $a(s,x) : [0,1] \times R^N \to R^d$ be measurable in (s,x) and continuously differentiable in x for each s. Suppose that $|\phi(x)| + \Sigma|\frac{\partial \phi}{\partial x_i}(x)| < K(1 + |x|^a)$ for some K, $a > 0$ if $\phi = f(s,x)$ or $a_i(s,x)$, $s < 1$. Let $x(t)$ be a $\underline{B}_t$-adapted process such that $x(t) \in H^{(\infty)}$, $t < 1$, $|Dx(t)|_H$ has a measurable version, and $E[\sup_{[0,1]} |x(t)|^p + |Dx(t)|^p] < \infty$ for each $p > 1$. Then if

$$v(t) = \int_0^t f(s,x(s))ds + \int_0^t \langle a(s,x(s)),db(s)\rangle$$

$v(t) \in H^{(\infty)}$ for each t, $0 < t < 1$. Also $E[\sup_{(0,1]} |Dv(t)|_H^p] = M_p < \infty$ and M_p depends only on p and $E[\sup_{[0,1]} |x(t)|^p + |Dx(t)|_H^p]$.

Proof. The proof is no different, except in notational complexity, if we assume $N = d = 1$. Consider the first term $z(t) = \int_0^t a(s,x(s))db(s)$. Note that for each s, $0 < s < 1$, $a(s,x(s)) \in H^{(\infty)}$ and $a(s,x(s))[b(t) - b(s)] \in H^{(\infty)}$ because of lemma (4.10). Furthermore $E[\sup_{[0,1]} |a(t,x(t))|^p + |Da(t,x(t))|_H^p] = K_p < \infty$ for each $p > 1$ where K_p depends only on $E[\sup_{[0,1]} |x(t)|^p + |Dx(t)|_H^p] < \infty$, K and a, for each $p > 1$. Now let $\{T^{(m)}\}$ be a sequence of partitions of $[0,1]$, and define

$$a^{(m)}(s,x(s)) = \sum_{t_i \in T^{(m)}} a(t_i,x(t_i))1_{(t_i,t_{i+1}]}(s) \quad .$$

Using the method in Doob [4], p. 440, choose $\{T^{(m)}\}$ so that $\lim_{m\to\infty} \int_0^1 E|a(s,x(s)) - a^{(m)}(s,x(s))|^2 ds = 0$. Clearly, from the above, it follows that

$$z^{(m)}(t) = \int_0^t a^{(m)}(s,x(s))db(s)$$

$$= \sum_{t_i \in T^{(m)}} a^{(m)}(t_i,x(t_i))[b(t_{i+1}\ t) - b(t_i\ t)]$$

is in $H^{(\infty)}$, and, for each $p > 1$, $\lim_{m\to\infty} E|z(t) - z^{(m)}(t)|^2 = 0$. Thus to conclude that $z(t) \in H^{(\infty)}$ is is sufficient to show that $\sup_m E|Dz^{(m)}(t)|_H^p < \infty$ for every p.. In the following, let $b(\Delta t_i) = b(t_{i+1}\wedge t) - b(t_i\wedge t)$. We want to study

$$\xi^{(m)}(t) = \langle Dz^{(m)}(t),\ Dz^{(m)}(t)\rangle_H$$

(4.13)
$$= \sum_{t_j,t_i \in T^{(m)}} \langle D[a(t_i,x(t_i))b(\Delta t_i)],\ D[a(t_j,x(t_j))b(\Delta t_j)_H\rangle \quad .$$

For each i, $D[a(t_i,x(t_i))b(\Delta t_i)] = a(t_i,x(t_i))Db(\Delta t_i) + (Da(t_i,x(t_i))b(\Delta t_i)$. Moreover, if $i < j$, corollary (2.9) implies that

$$\langle Da(t_i,x(t_i)),\ Db(\Delta t_j)\rangle_H = 0 \quad .$$

Likewise, it is easy to compute $\langle Db(\Delta t_i),\ b(\Delta t_j)\rangle_H = (t_{i+1}\wedge t - t_i\wedge t)\delta_{ij}$. By applying these identities, the individual term of (4.13) becomes, if $i < j$,

$$\langle D[a(t_i,x(t_i))b(\Delta t_i)],\ D[a(t_j,x(t_j))b(\Delta t_j)]\rangle_H$$

$$= \langle D[a(t_i,x(t_i))b(\Delta t_i)],\ Da(t_j,x(t_j))\rangle_H\ b(\Delta t_j)$$

$$+ \langle Da(t_i,x(t_i)),\ Da(t_j,x(t_j))\rangle_H\ (t_{i+1}\ t - t_i\ t)\delta_{ij} \quad .$$

Summing and rearranging these terms in (4.13) gives

$$\xi^{(m)}(t) = 2\int_0^t \langle Dz^{(m)}(s),\ Da^{(m)}(s)\rangle_H db(s)$$

$$+ \int_0^t [|Da^{(m)}(s)|_H^2 + |a^{(m)}(s)|_H^2] ds \quad .$$

It follows from the Burkholder-Gundy inequality that

$$E|\xi^{(m)}(t)|^{p/2} \leq K_p' \int_0^t E|\langle Dz^{(m)}(s),\ Da^{(m)}(s)\rangle_H|^{p/2} ds$$

$$+ K_p' t^{p-1} \int_0^t [|Da^{(m)}(s)|_H^p + |a^m(s)|^p] ds$$

$$\leq K_p'' \int_0^t (1 + E|\xi^{(m)}(s)|^{p/2}) ds + K_p'' t^p$$

$$= K_p'' \int_0^t E|\xi^m(s)|^{p/2} ds + K_p''(t+t^p)$$

where K_p' and K_p'' depend only on p and K_p independently of m. By the Gronwall-Bellman inequality $\sup_m E|Dz^{(m)}(t)|^p = \sup_m E|\xi^m(t)|^{p/2} < \infty$. This completes the proof for $z(t)$, and the term $\int_0^t f(s,x(s))ds$ is treated similarly.

□

These results prepare us for proving the weak H-differentiability of solutions to (4.1).

(4.14) <u>Theorem</u>. Let $y(t)$ be the solution of (4.1) and assume that conditions (4.2) and (4.3) are satisfied. Then $y(t) \in H^{(\infty)}$ for all $t \in [0,1]$ and

(4.15) $$Dy(t)(\tau) = \int_0^{\tau\wedge t} Z(t)Z^{-1}(s)\sigma(s,y(s))ds \quad .$$

We first prove

(4.16) <u>Lemma</u>. Assume in addition to the hypotheses of theorem (4.14) that m and σ are twice continuously diferentiable in y and that the second derivatives are slowing growing functions. Then $y(t) \in H^{(\infty)}$ and (4.15) is valid.

<u>Proof</u>. Consider the usual Picard iteration

$$y^{(0)}(t) \equiv y_0$$

$$y^{(n+1)}(t) = y_0 + \int_0^t m(s,y^{(n)}(s))ds + \int_0^t \sigma(s,y^{(n)}(s))db(s) \quad .$$

Stroock [9] shows that for each n, $y^n(t) \in D_p$, for all t and for all $p > 1$, and

$|Dy^{(n)}(t)|_H$ has a measurable version. Furthermore,

$$\lim_{n\to\infty} E \sup_{[0,1]} |y(t) - y^{(n)}(t)|^p = 0 \qquad p > 1$$

and $\sup_n E[\sup_{[0,1]} (|y^{(n)}(t)|^p + |Ay^{(n)}(t)|^p + |Dy^{(n)}(t)|_H^p)] < \infty$. Lemma 4.12 implies that $y^{(n)}(t) \in H^{(\infty)}$ for each n. From these observations it follows that $y(t) \in H^{(\infty)}$, as desired.

It remains to prove (4.15) under the added assumptions of the lemma. Let $h \in H^{(\infty)}$,

$$\eta(t) = \int_0^t \langle h'(s), db(s)\rangle \quad \text{and} \quad \xi_k(t) = \langle Dy_k(t), h\rangle_H \quad .$$

Then, using Stroock's [9] application of Malliavin calculus to stochastic d.e.'s

$$\begin{aligned}
\xi_k(t) &= \langle Dy_k(t),\ D\eta(t)\rangle_H \\
&= \nabla y_k(t)\cdot\nabla\eta(t) \\
&= \int_0^t \sum_{\ell=1}^N \frac{\partial m_k}{\partial y_\ell}(s,y(s))\xi_\ell(s)ds \\
&\quad + \int_0^t \sum_{\ell=1}^N \xi_\ell(s) \sum_{j=1}^d \frac{\partial \sigma_{kj}}{\partial y_\ell}(s,y(s))db_j(s) \\
&\quad + \int_0^t \sum_{j=1}^d \sigma_{kj}(s)h_j'(s)ds \quad .
\end{aligned}$$

In other words

$$d\xi(t) = \partial_y m(t,y(t))\cdot\xi(t)dt + \sum_{j=1}^d \partial_y \sigma_j(t);\xi(t)db_j(t)$$

$$+ \sigma(t,y(t))h'(t)dt$$

$$\xi(0) = 0 \quad .$$

The solution of this equation is precisely

$$\xi(t) = \int_0^t Z(t)Z^{-1}(s)\sigma(s,y(s))h'(s)ds \quad ,$$

and $Dy(t)(\tau) = \int_0^{t\wedge\tau} Z(t)Z^{-1}(s)\sigma(s,y(s))ds$ follows directly.

Proof of theorem (4.14). Assume m and σ satisfy (4.2) - (4.3). Let $\rho \in C_0^\infty(\mathbb{R}^N)$ such that $\int \rho dx = 1$ and $\rho \geq 0$, and let $\rho_n(x) = n^d\rho(nx)$. Define

$$m^{(n)}(t,y) = (\rho_n * m(t,\cdot))(y)$$

$$\sigma^{(n)}(t,y) = (\rho_n * \sigma(t,\cdot))(y)$$

$$dy_n(t) = m^{(n)}(t,y_n)dt + \sigma^{(n)}(t,y_n)d\beta$$

$$y_n(0) = 0$$

$$dZ_n = \partial_y m^{(n)}(t,y_n)\cdot Z_n dt + \sum_{i=1}^{d} \partial_y \sigma_i^{(n)}(t,y_n)\cdot Z_n db_i(t) \tag{4.17}$$

$$Z_n(0) = I \quad .$$

It may be shown that $m^{(n)} \to m$ and $\sigma^{(n)} \to \sigma$ uniformly on compacts as $n \to \infty$. Moreover, for each n, $m^{(n)}$ and $\sigma^{(n)}$ satisfy the hypotheses of lemma 4.12, and, there is a constant K' such that

$$\begin{aligned} &|m^{(n)}(t,y)| + \|\sigma^{(n)}(t,y)\| \leq K'(1 + |y|), \text{ and} \\ &|m^{(n)}(t,y) - m^{(n)}(t,x)| + \|\sigma^{(n)}(t,x) - \sigma^{(n)}(t,y)\| \\ &\qquad \leq K'|x-y| \end{aligned} \tag{4.18}$$

for all $0 \leq t \leq 1$, for every n. These facts provide a sufficient basis to prove, with a standard, Gronwall-Bellman inequality argument,

$$\lim_{n\to\infty} E\{\sup_{0\leq t\leq 1} |y_n(t) - y(t)|^p\} = 0 \tag{4.19}$$

for every $p > 1$.

By construction of $m^{(n)}$ and $\sigma^{(n)}$ and lemma 4.12, $y_n(t) \in H^{(\infty)}$ for each n. Thus, proving

$$\lim_{n\to\infty} E|Dy_n(t) - \gamma|_H^p = 0 \tag{4.20}$$

for every $p > 1$, where $\gamma(\tau) = \int_0^{t\wedge\tau} Z(t)Z^{-1}(s)\sigma(s,y(s))ds$ is sufficient to prove that $y(t) \in H^\infty$ and $Dy(t)$ is given by (4.15). However, it is easy to see that (4.20) holds if (4.19) is true and

$$\lim_{n\to\infty} E[\sup_{0\leq t\leq 1} \|Z_n(t) - Z(t)\|_2^p] = 0 \tag{4.21}$$

$$(4.22) \qquad \lim_{n\to\infty} E[\sup_{0\le t\le 1} \|Z_n^{-1}(t) - Z^{-1}(t)\|_2^p] = 0$$

for every p. ($\|Z\|_2^2 = \Sigma |Z_{ij}|^2$). Thus, to complete the proof we need only verify these limits.

An easy extension of theorem 5.2 in [12] shows that (4.18), (4.2), (4.3) and the condition (4.23 below are enough to guarantee (4.21):

$$\forall t \le 1,\ \varepsilon > 0,\ M < \infty$$

$$(4.23) \qquad \lim_{n\to\infty} P(\sup_{\|Z\|_2\le M} \|[\partial_y m^{(n)}(t,y_n) - \partial_y m(t,y)]\cdot Z\| > \varepsilon) = 0$$

$$\lim_{n\to\infty} P(\sup_{\|Z\|_2\le M} \|\sum_{i=1}^{d} [\partial_y \sigma_i^{(n)}(t,y_n) - \partial_y \sigma_i(t,y)]\cdot Z\| > \varepsilon) = 0 \quad .$$

Since $\partial_y m(t,y)$ is continuous in y, $\partial_y m(t,y_n) \overset{P}{\to} \partial_y m(t,y)$ as $n \to \infty$ for each t, and, since $\partial_y m^{(n)} \to \partial_y m$ uniformly on compacts as $n \to \infty$, and $\sup_n E|y_0(t)|^2 < \infty$, $\partial_y m^{(n)}(t,y_n) \overset{P}{\to} \partial_y m(t,y)$ as $n \to \infty$ for each t. This proves that $\partial_y m^{(n)}(t,y_n) \overset{P}{\to} \partial_y m(t,y)$, $n \to \infty$, which is enough to verify the first limit in (4.23). Completely analogous arguments demonstrate the limits in (4.23) involving σ_i. This completes the proof of (4.21). Since Z^{-1} satisfies equation (4.6), and $Z_n^{-1}(t)$ the analogous equation with m and σ replaced by $m^{(n)}$ and $\sigma^{(n)}$, (4.23) and the additional condition

$$\lim_{n\to\infty} P(\sup_{|Z|\le M} \|Z(\sum_{i=1}^{d} [\partial_y \sigma^{(n)}(t,y_n)]^2 - [\partial_y \sigma(t,y)]^2\|_2 > \varepsilon) = 0$$

for every $0 \le t \le 1$, $\varepsilon > 0$ and $M < \infty$, suffice to imply (4.22). But, this last condition is true by repeating the arguments from (4.23). This completes the proof.

□

<u>Proof of theorem 4.7</u>. First suppose that the theorem is true if, in addition to satisfying the hypothesis of theorem 4.7, F is a cylinder function $F(x) = \phi(x(t_1),\dots,x(t_n))$. In this case, note that (4.8) implies ϕ is C^1 and

$$(4.24) \qquad |\phi(x)| + |\sum_1^n \frac{\partial\phi}{\partial y_i}(x)| \le K(1 + \|x\|^a) \quad .$$

To prove the general case, let $\{T^{(n)}\}$ be a sequence of partitions of $[0,1]$ and define $x^{(n)}$ for $x \in B$ as in the proof of lemma 2.8. Likewise, define $F_n(x) = F(x^{(n)})$. Then if F satisfies (4.8), $\lim_{n\to\infty} F_n(y(\cdot,b)) = F(y(\cdot,b))$ a.s. and there exists a subsequence n_k such that

$$
\begin{aligned}
(4.25)\qquad &\lim_{k\to\infty} \int_0^1 E\{\int_{(s,1]} \lambda^{F_{n_k}}(ds;y)Z(t)Z^{-1}(s)\sigma(s,y(s))|a_s\}db(s) \\
&= \int_0^1 E\{\int_{(s,1]} \lambda^{F}(ds;y)Z(t)Z^{-1}(s)\sigma(s,y(s))|a_s\}db(s) \quad .
\end{aligned}
$$

A nice proof of (4.25) is given in Davis [3] and will be omitted here. Now if (4.7) is true for cylinder functions

$$
(4.26)\qquad F_n(y(\cdot,\beta)) = \int_0^1 E\{\int_{(s,1]} \lambda^{F_n}(ds;y)Z(t)Z^{-1}(s)\sigma(s,y(s))|\underline{B}_s\}db(s) \quad ,
$$

using (4.25) to take limits, we find that this equation is true if F_n is replaced by any F satisfying (4.8). Since (4.26) and (4.9) are equivalent, this completes the proof once the cylinder function case is established.

Thus let

$$
F(y) = \phi(y(t_1),\ldots,y(t_n)) = \phi(\bar{y}) \quad .
$$

Since $y(t_i) \in H^{(\infty)}$ for $1 \le i \le t_n$, and since ϕ satisfies (4.24), we find from lemma (4.10) and (4.15) that $F(y) \in H^{(\infty)}$ and

$$
\begin{aligned}
DF(y)'(s) &= \sum_{i=1}^{n} \frac{\partial\phi}{\partial y_i}(\bar{y})Z(t_i)Z^{-1}(s)\sigma(s,y(s))1_{\{s<t_i\}} \\
&= \int_{(s,1]} \lambda^{F}(du;y)Z(u)Z^{-1}(s)\sigma(s,y(s)) \quad .
\end{aligned}
$$

For such F, (4.9), and hence (4.26), are direct consequences of Clark's formula (3.1).

□

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If F is a Fréchet differentiable functional on $C[0,1]$, $b = \{b(t)|0 \le t \le 1\}$ is a Brownian motion, and $\underline{B}_t = \sigma\{b(s)|s \le t\}$, Clark's formula states that $F(b) = \int_0^1 E\{\lambda^F(s,1];\cdot)|\underline{B}_s\}db(s)$, where $\lambda^F(du;b)$ is the measure defining the Fréchet derivative of F at b. In this paper we extend Clark's formula to the more general class of weakly H-differentiable functionals, and we give a simple proof based on Malliavin's calculus. Again using Malliavin calculus techniques, we also derive Haussmann's stochastic integral

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ABSTRACT (continued)

representation of a functional $F(y)$ of the diffusion process $dy = m(t,y)dt + \sigma(t,y)db$. In doing this, we show that $y(t)$ is weakly H-differentiable if m and σ have bounded, continuous, first derivatives in y.

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